INSTITUTO DE FÍSICA

Universidade de São Paulo

Eletromagnetismo II

6ª Aula

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Na última aula vimos...

- Ondas em *meios dielétricos*:

$$\begin{cases} i)\ \text{Índice de refração: } n = \sqrt{\varepsilon_R} \\ ii)\ \text{Número de onda: } k = \sqrt{\varepsilon_R}\,\dfrac{\omega}{\mathrm{c}} \end{cases}$$

- Sendo: $\boxed{\varepsilon_R = \dfrac{\varepsilon}{\varepsilon_0}}$ (*c*te. dielétrica)

- Cálculo dos campos a partir dos potenciais $\vec{A}$ e φ:

$$\begin{cases} \vec{B} = \vec{\nabla} \times \vec{A} & \textbf{(1)} \\ \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t} & \textbf{(2)} \end{cases}$$

- E, para as *Eqs. de Maxwell* serem satisfeitas:

$$\begin{cases} -\nabla^2\varphi - \dfrac{\partial\left(\vec{\nabla}\cdot\vec{A}\right)}{\partial t} = \dfrac{\rho}{\varepsilon} & (3) \\ -\nabla^2\vec{A} + \varepsilon\mu\dfrac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \varepsilon\mu\dfrac{\partial\varphi}{\partial t}\right) = \mu\vec{J} & (4) \end{cases}$$

- $\vec{A}$ e φ não definem univocamente $\vec{E}$ e $\vec{B}$ $\Rightarrow$ podemos escolher "calibres" (*gauges*) adequados.

- Para ondas EM, o de *Lorentz* :

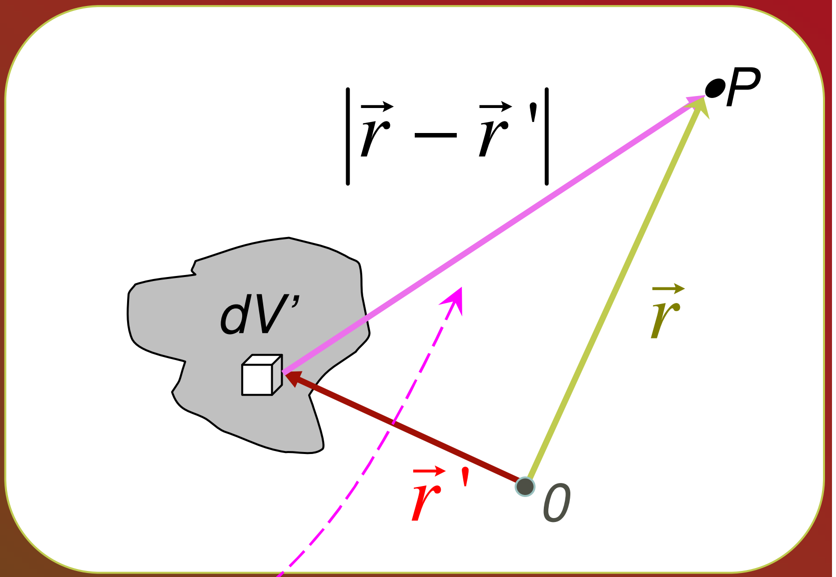
$$\left(\vec{\nabla}\cdot\vec{A} = -\varepsilon\mu\frac{\partial\varphi}{\partial t}\right)$$

$$\therefore \quad \begin{cases} -\nabla^2\varphi + \varepsilon\mu\dfrac{\partial^2\varphi}{\partial t^2} = \dfrac{\rho}{\varepsilon} \\ -\nabla^2\vec{A} + \varepsilon\mu\dfrac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J} \end{cases}$$

equações desacopladas!

- Discutimos também *maneira alternativa* para se obter φ e $\vec{A}$:

$$\begin{cases} \vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \dfrac{\vec{J}\left(\vec{r}\,',t_r\right)}{\left|\vec{r}-\vec{r}\,'\right|}dV' \\ \varphi(\vec{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho\left(\vec{r}\,',t_r\right)}{\left|\vec{r}-\vec{r}\,'\right|}dV' \end{cases}$$



- Que correspondem aos *potenciais retardados*:

- *Tempo retardado* : $\boxed{t_r = t - \dfrac{\left|\vec{r}-\vec{r}\,'\right|}{c}}$; supondo o ponto *P* situado no vácuo.

- Vamos mostrar agora que as expressões dos potenciais retardados, de fato, *são soluções* das equações diferencias desacopladas.

- Calculando o *grad.* do *potencial escalar*:

$$\left(\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\vec{r}\,',t_r\right)}{\left|\vec{r}-\vec{r}\,'\right|}\,dV\,' \right)$$

$$\vec{\nabla}\varphi = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\vec{\nabla}\rho}{\left|\vec{r}-\vec{r}\,'\right|} + \rho\,\vec{\nabla}\left(\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\right)\right] dV\,'$$

vou chamar $\boxed{\left|\vec{r}-\vec{r}\,'\right| = R}$

- Sendo que: $\vec{\nabla}\rho = \vec{\nabla}\rho\left(\vec{r}\,',t_r\right) = \vec{\nabla}\rho\left(\vec{r}\,',t-\frac{R}{c}\right)$

aplicado às coordenadas sem linha

- Agora:

$$\vec{\nabla}\rho = \frac{\partial\rho}{\partial x}\vec{i} + \frac{\partial\rho}{\partial y}\vec{j} + \frac{\partial\rho}{\partial z}\vec{k} = \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial x}\vec{i} + \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial y}\vec{j} + \frac{\partial\rho}{\partial t_r}\frac{\partial t_r}{\partial z}\vec{k}$$

$$\boxed{\frac{\partial\rho}{\partial t_r} = \dot{\rho}} \; ;\{\rho = \rho(\vec{r}\,', t_r)\}$$

- Assim:

$$\vec{\nabla}\rho = \dot{\rho}\left(\frac{\partial t_r}{\partial x}\vec{i} + \frac{\partial t_r}{\partial y}\vec{j} + \frac{\partial t_r}{\partial z}\vec{k}\right) = \dot{\rho}\,\vec{\nabla}t_r$$

- Mas: $\vec{\nabla}t_r = \vec{\nabla}\left(t - \frac{R}{c}\right) = -\frac{1}{c}\vec{\nabla}R = -\frac{1}{c}\vec{\nabla}\left|\vec{r} - \vec{r}\,'\right|$

- Sendo que:

$\longrightarrow$ aplicado nas coords. SEM LINHA

$$\vec{\nabla}\left|\vec{r}-\vec{r}\,'\right|=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\left\{\underbrace{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}_{\equiv\ \left|\vec{r}-\vec{r}\,'\right|=R}\right\}$$

- Derivando:

$$\vec{\nabla}\left|\vec{r}-\vec{r}\,'\right|=\frac{1}{\cancel{2}}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\times$$

$$\times\ \cancel{2}\left((x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}\right)$$

- Então:

$$\boxed{\vec{\nabla}\left|\vec{r}-\vec{r}\,'\right|}=\frac{\left((x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}=\frac{\vec{R}}{R}\ \boxed{=\hat{R}}$$

- Seguindo o mesmo procedimento: $\vec{\nabla}\left(\dfrac{1}{R}\right) = -\dfrac{\hat{R}}{R^2}$

- Lembrando os resultados já obtidos:

$$\boxed{\vec{\nabla}\rho = \dot{\rho}\,\vec{\nabla}t_r}\;;\;\boxed{\vec{\nabla}t_r = -\frac{1}{c}\vec{\nabla}R = -\frac{1}{c}\vec{\nabla}\left|\vec{r}-\vec{r}\,'\right|}\;\; e \;\;\boxed{\vec{\nabla}\left|\vec{r}-\vec{r}\,'\right| = \hat{R}}$$

- Então:

$$\vec{\nabla}\varphi = \frac{1}{4\pi\varepsilon_0}\int_V\left[\frac{\vec{\nabla}\rho}{\underbrace{\left|\vec{r}-\vec{r}\,'\right|}_{R}} + \rho\,\vec{\nabla}\left(\frac{1}{\underbrace{\left|\vec{r}-\vec{r}\,'\right|}_{R}}\right)\right]dV\,' =$$

$$= \frac{1}{4\pi\varepsilon_0}\int_V\left[-\frac{1}{c}\frac{\dot{\rho}}{R}\hat{R} - \rho\frac{\hat{R}}{R^2}\right]dV\,'$$

- Reescrevendo:

$$\begin{cases} \varphi(\vec{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV' \\ \Downarrow \\ \vec{\nabla}\varphi = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \left[-\dfrac{1}{c}\dot{\rho}\underline{\dfrac{\hat{R}}{R}} - \rho\underline{\dfrac{\hat{R}}{R^2}} \right] dV' \end{cases}$$

- Lembrando da *eq. diferencial*: $\left(-\nabla^2\varphi + \varepsilon\mu\dfrac{\partial^2\varphi}{\partial t^2} = \dfrac{\rho}{\varepsilon} \right)$

- Vou agora calcular $\vec{\nabla}\cdot\left(\vec{\nabla}\varphi\right) = \nabla^2\varphi$

- Usando que: $\vec{\nabla}\cdot\left(g\vec{F}\right) = g\vec{\nabla}\cdot\vec{F} + \left(\vec{\nabla}g\right)\cdot\vec{F}\ ;\begin{cases} \left(\mathrm{g} = \rho, \dot{\rho}\right) \\ \left(\vec{\mathrm{F}} = \dfrac{\hat{\mathrm{R}}}{\mathrm{R}^2}, \dfrac{\hat{\mathrm{R}}}{\mathrm{R}}\right) \end{cases}$

- Assim:

$$\nabla^2\varphi = \frac{1}{4\pi\varepsilon_0}\int_V \left\{ -\frac{1}{c}\left[\dot{\rho}\underbrace{\left(\vec{\nabla}\cdot\frac{\hat{R}}{R}\right)}_{(\mathrm{i})} + \frac{\hat{R}}{R}\cdot\underbrace{\left(\vec{\nabla}\dot{\rho}\right)}_{(\mathrm{ii})}\right] - \left[\rho\underbrace{\left(\vec{\nabla}\cdot\frac{\hat{R}}{R^2}\right)}_{(\mathrm{iii})} + \frac{\hat{R}}{R^2}\cdot\underbrace{\vec{\nabla}\rho}_{(\mathrm{iv})}\right]\right\} dV'$$

(i) $\quad \vec{\nabla}\cdot\frac{\hat{R}}{R} = \vec{\nabla}\cdot\frac{\vec{R}}{R^2} =$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\left(x^2+y^2+z^2\right)} = \frac{1}{\left(x^2+y^2+z^2\right)} - \frac{2x^2}{\left(x^2+y^2+z^2\right)^2} + \ldots$$

$$= \frac{3}{\left(x^2+y^2+z^2\right)} - \frac{2\cancel{\left(x^2+y^2+z^2\right)}}{\left(x^2+y^2+z^2\right)^{\cancel{2}}}$$

Então: $\boxed{\vec{\nabla}\cdot\frac{\hat{R}}{R} = \frac{1}{R^2}}$

(ii) $\vec{\nabla}\dot{\rho}$ (havíamos obtido: $\vec{\nabla}\rho = \dot{\rho}\,\vec{\nabla}t_r$; $\vec{\nabla}t_r = -\frac{1}{c}\vec{\nabla}R$)

- Da mesma forma:

$$\underline{\underline{\vec{\nabla}\dot{\rho}}} = \ddot{\rho}\,\vec{\nabla}t_r = \ddot{\rho}\left(-\frac{1}{c}\vec{\nabla}R\right) = -\frac{1}{c}\underline{\underline{\ddot{\rho}\,\hat{R}}} \quad ; \quad \ddot{\rho} = \frac{\partial^2\rho}{\partial t_r^2}$$

(iii) $\vec{\nabla}\cdot\frac{\hat{R}}{R^2} = 4\pi\,\delta^3(\vec{R})$; $\delta^3(\vec{R}) \equiv$ *Delta de Dirac*

<u>*Propriedades*</u>:

$$\left.\begin{array}{l} \int f(\vec{r})\,\delta^3(\vec{r}-\vec{r}_0)\,dV = f(\vec{r}_0) \\ \int \delta^3(\vec{r})\,dV = 1 \end{array}\right\}$$

<u>Em todo o espaço</u>

$$(\text{iv})\ \vec{\nabla}\rho = (como\ ja'\ visto) = -\frac{1}{c}\dot{\rho}\vec{\nabla}R = -\frac{1}{c}\dot{\rho}\hat{R}$$

- Desta forma:

$$(\hat{R}\cdot\hat{R}=1)$$

$$\nabla^2\varphi = \frac{1}{4\pi\varepsilon_0}\left\{\int \cancel{(-)\frac{1}{c}\frac{\dot{\rho}}{R^2}}dV' - \frac{1}{c}\int\left(-\frac{1}{c}\right)\frac{\hat{R}}{R}\cdot\ddot{\rho}\hat{R}\,dV' - \right.$$

$$\left. -\int 4\pi\,\rho\,\delta^3(\vec{R})\,dV' - \int \cancel{\frac{\hat{R}}{R^2}\cdot\left(-\frac{1}{c}\right)\dot{\rho}\hat{R}\,dV'}\right\}$$

- E também:

$$\int 4\pi\,\rho\,\delta^3(\vec{R})\,dV' = 4\pi\int\rho\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)\delta^3(\vec{r}-\vec{r}\,')\,dV'$$

$$(\text{substituo } \vec{\mathrm{r}}\,'\to\vec{\mathrm{r}})$$

$$\therefore\quad \boxed{4\pi\int\rho\,\delta^3(\vec{R})\,dV' = 4\pi\,\rho(\vec{r},t)}$$

- Ou : $$\nabla^2\varphi = \frac{1}{c^2}\frac{\partial^2}{\partial t_r^2}\underbrace{\left(\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}dV'\right)}_{\equiv\varphi(\vec{r},t)} - \frac{\rho(\vec{r},t)}{\varepsilon_0}$$

- Então: $$\boxed{\nabla^2\varphi(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t_r^2}\varphi(\vec{r},t) - \frac{\rho(\vec{r},t)}{\varepsilon_0}}$$

- Sendo:

$$\boxed{\frac{\partial}{\partial t_r}} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t_r} \boxed{= \frac{\partial}{\partial t}} \quad ;\, pois \quad \frac{\partial t_r}{\partial t} = \frac{\partial}{\partial t}\left(t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right) = \frac{\partial t}{\partial t} = 1$$

- Que corresponde à eq. diferencial original! Portanto $\varphi(\vec{r},t)$ é, de fato, solução!

- Mesmo procedimento é utilizado para mostrar que $\vec{A}$ também satisfaz a equação diferencial correspondente (posso trabalhar com A_x, A_y, A_z, se for conveniente).

- Um fato interessante. Se considerarmos os '***Potenciais Avançados***' :

$$\varphi_a(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,', t_a)}{|\vec{r}-\vec{r}\,'|}dV' \quad e \quad \vec{A}_a(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,', t_a)}{|\vec{r}-\vec{r}\,'|}dV'$$

$$\boxed{t_a = t + \frac{|\vec{r}-\vec{r}\,'|}{c}}$$

- Pode-se mostrar que as Equações de Maxwell (gauge de Lorentz), são satisfeitas, mas violariam o "***princípio da causalidade***" (todo efeito tem uma causa).

Na próxima aula veremos …

Propagação de Ondas EM